

Topics Concerning the Quadrupole-Quadrupole Interaction

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Abstract

We address some properties of the quadrupole-quadrupole ($Q \cdot Q$) interaction in nuclear studies. We first consider how to restore $SU(3)$ symmetry even though we use only coordinate and not momentum terms. Using the Hamiltonian $H = \sum_i (\frac{p_i^2}{2m} + \frac{m}{2} \omega^2 r_i^2) - \chi \sum_{i < j} Q(i) \cdot Q(j) - \frac{\chi}{2} \sum_i Q(i) \cdot Q(i)$ with $Q_\mu = r^2 Y_{2,\mu}$, we find that only 2/3 of the single-particle splitting ($\epsilon_{0d} - \epsilon_{1s}$) comes from the diagonal term of $Q \cdot Q$ -the remaining 1/3 comes from the interaction of the valence nucleus with the core. On another topic, a previously derived relation, using $Q \cdot Q$, between isovector orbital $B(M1)$ (scissors mode) and the *difference* ($B(E2, isoscalar) - B(E2, isovector)$) is discussed. It is shown that one needs the isovector $B(E2)$ in order that one get the correct limit as one goes to nuclei sufficiently far from stability so that one subshell (neutron or proton) is closed.

In this work we address issues pertaining to shell model calculations with the schematic quadrupole-quadrupole interaction. Even today, this interaction is of value in casting light upon the relationship between shell model and collective model behaviour. There are still

new things to be learnt about this interaction in nuclei, and we will discuss two examples here.

I. THE SINGLE-PARTICLE SPLITTING ($\epsilon_{0D} - \epsilon_{1S}$) NEEDED TO GET THE $SU(3)$ RESULT

We wish to obtain Elliott's $SU(3)$ results [1] in a shell model calculation in which only the coordinate $Q \cdot Q$ interaction is used. We do not wish to use the momentum-dependent terms. The latter were introduced by Elliott so that, in combination with the coordinate terms, there would be no $\Delta N = 2$ admixtures i.e. no admixture from configurations involving 2 $\hbar\omega$ excitations. However, we *want* to see the effects of such admixtures in our shell model studies. One classic problem in which $\Delta N = 2$ admixtures are important is the $E2$ effective charge, but there are many other problems of interest along these lines.

The Hamiltonian we consider is therefore

$$H = \sum_i \left(\frac{p^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) - \chi \sum_{i < j} Q(i) \cdot Q(j) - \frac{\chi}{2} \sum_i Q(i) \cdot Q(i)$$

where $Q(i)^k \cdot Q(j)^k = (-1)^k \sqrt{2k+1} r(i)^k r(j)^k [Y(i)^k Y(j)^k]^0$ with $k = 2$. Like Elliott, we have not only the two-body $Q \cdot Q$ term, but also the $i = j$ single-particle term.

It is convenient to introduce the following quantity: $\bar{\chi} = 5b^4\chi/32\pi$ where b is the oscillator length parameter, such that $b^2 = \hbar/m\omega = 41.46/\hbar\omega$.

To evaluate the single-particle term we use the addition theorem:

$$P_k(\cos\theta_{12}) = \frac{4\pi}{2k+1} \sum_{\mu} Y_{k,\mu}(1) Y_{k,-\mu}(2)$$

thus

$$\sqrt{5}[Y^2(i)Y^2(i)]^0 = \frac{5}{4\pi}P_2(1) = \frac{5}{4\pi}$$

The single-particle potential is then

$$U(r) = -\frac{\chi}{2}Q(i) \cdot Q(i) = -4\bar{\chi} \left(\frac{r}{b} \right)^4$$

The expectation value of this single-particle term for various single-particle states is given in Table I. What single-particle splitting $\epsilon_{0d} - \epsilon_{1s}$ is needed to get Elliott's $SU(3)$ results? The best way to answer this is to give the formula for the $SU(3)$ energy in the $1s - 0d$ shell (in which the momentum terms are included):

$$E(\lambda\mu) = \bar{\chi} \left[-4(\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)) + 3L(L + 1) \right]$$

For a rotational band, the $L = 2 - L = 0$ splitting is given by the last term and is equal to $18\bar{\chi}$. This must also be the $\epsilon_{0d} - \epsilon_{1s}$ because it is also an $L = 2 - L = 0$ splitting. But, as seen from Table I, the splitting due to the diagonal $Q \cdot Q$ interaction is $(-63 - (-75))\bar{\chi} = 12\bar{\chi}$. Where does the remaining $6\bar{\chi}$ come from?

The answer is that the missing part comes from the interaction of the particle with the core. For $Q \cdot Q$, the only contribution is the exchange term of the $0d$ particle with the $0s$ core.

Thus, to get the Elliott $SU(3)$ results in the $1s - 0d$ shell, we must not only include his diagonal term but also include the particle-core interaction *i.e.* take the shell model as an A particle problem rather than an $(A - 16)$ particle problem.

The same thing happens in the $0f - 1p$ shell. The single-particle splitting required to get the $SU(3)$ result is $\epsilon_{0f} - \epsilon_{1p} = 3(3 \times 4 - 1 \times 2)\bar{\chi} = 30\bar{\chi}$. As seen from Table I, we only get $2/3$ of this ($20\bar{\chi}$) from the diagonal $Q \cdot Q$ term. The remaining $10\bar{\chi}$ comes from the interaction of the valence nucleons with the core (actually only the $0p$ shell in the core will contribute).

II. CLARIFICATION OF A RELATION BETWEEN THE ISOVECTOR ORBITAL MAGNETIC DIPOLE TRANSITION RATE (I.E. SCISSORS MODE EXCITATION RATE) AND THE ELECTRIC QUADRUPOLE TRANSITION.

As a second example, we will attempt to clarify a relationship between orbital magnetic dipole transition rates (i.e. scissors mode excitation rates) and electric quadrupole transitions. Using the interaction $-\chi Q \cdot Q$, Zheng and Zamick [4] obtained a sum rule relating

these two quantities. The isovector orbital magnetic dipole operator is $(\vec{L}_\pi - \vec{L}_\nu)/2$ (the isoscalar one is half the total orbital angular momentum $\vec{L}/2 = (\vec{L}_\pi + \vec{L}_\nu)/2$). In detail, the sum rule reads

$$\sum_n (E_n - E_0) B(M1)_o = \frac{9\chi}{16\pi} \sum_i \{ [B(E2, 0_1 \rightarrow 2_i)_{IS} - B(E2, 0_1 \rightarrow 2_i)_{IV}] \}$$

where $B(M1)_o$ is the value for the *isovector* orbital $M1$ operator ($g_{l\pi} = 0.5$ $g_{l\nu} = -0.5$ $g_{s\pi} = 0$ $g_{s\nu} = 0$) and the operator for the $E2$ transitions is $\sum_{protons} e_p r^2 Y_2 + \sum_{neutrons} e_n r^2 Y_2$ with $e_p = 1$, $e_n = 1$ for the isoscalar transition (IS), and $e_p = 1$, $e_n = -1$ for the isovector transition (IV). The above result holds also if we add a pairing interaction between like particles *i.e.* between two neutrons and between two protons.

The above work was motivated by the realization from many sources that there should be a relation between the scissors mode excitation rate and nuclear collectivity. Indeed, the initial picture by Palumbo and LoIudice [5] was of an excitation in a deformed nucleus in which the symmetry axis of the neutrons vibrated against that of the protons. In a 1990 contribution by the Darmstadt group [6], it was noted that the Sm isotopes, which undergo large changes in deformation, the $B(M1)_{scissors}$ was proportional to $B(E2, 0_1 \rightarrow 2_1)$. The $B(E2)$ in turn is proportional to the square of the nuclear deformation δ^2 .

The above energy-weighted sum rule of Zheng and Zamick was an attempt to obtain such a relationship microscopically using fermions rather than interacting bosons. To a large extent they succeeded, but there are some differences. Rather than the proportionality factor $B(E2, 0_1 \rightarrow 2_1)$, there is the difference of the isoscalar and isovector $B(E2)$. Now one generally expects the isoscalar $E2$ state to be most collective and much larger than the isovector $B(E2)$. If the latter is negligible, then indeed one basically has the same relation between scissors mode excitations and nuclear collectivity, as empirically observed in the Sm isotopes.

However, derivation of the above energy-weighted sum rule is quite general, and should

therefore hold (in the mathematical sense) in all regions -not just where the deformation is strong. To best illustrate the need for the isovector $B(E2)$, consider a nucleus with a closed shell of neutrons or protons. In such a nucleus, and neglecting ground-state correlations, the scissors mode excitation rate will vanish -one needs both open shell neutrons and protons to get a finite scissors mode excitation rate. On the other hand, the $B(E2, 0_1 \rightarrow 2_1)$ can be quite large. However, if we have say an open shell of protons and a closed shell of neutrons, the $B(E2, 0_1 \rightarrow 2_1)$ can be quite substantial. Many vibrational nuclei are of such an ilk, and they have large $B(E2)$'s from ground *e.g.* 20 W.u.

However, in the above circumstances, the neutrons will not contribute to the $B(E2)$ even if we give them an effective charge. But if only the protons contribute, it is clear that $B(E2, \text{isovector}) = B(E2, \text{isoscalar})$.

As an example, let us consider the even-even *Be* isotopes ^6Be , ^8Be , ^{10}Be and ^{12}Be . In so doing, we go far away from the valley of stability, but this is in tune with modern interests in radioactive beams.

Fayache, Sharma and Zamick [3] have previously considered ^8Be and ^{10}Be . The point was made that these two nuclei had about the same calculated $B(E2, 0_1 \rightarrow 2_1)$, but the isovector orbital $B(M1)$'s were significantly smaller in ^{10}Be than in ^8Be . This was against the systematic that $B(M1)_{\text{orbital}}$ is proportional merely to $B(E2)$. In detail, the calculated $B(M1, 0_1 \rightarrow 1_1)$ was $2/\pi\mu_N^2$ for ^8Be , and in ^{10}Be it was $9/32\pi\mu_N^2$ ($T = 1 \rightarrow T = 1$) and $15/32\pi\mu_N^2$ ($T = 1 \rightarrow T = 2$). Thus the ratio of isovector orbital $B(M1)$'s $^{10}\text{Be}/^8\text{Be} = 3/8$.

We now extend the calculations to include ^6Be and ^{12}Be . These are singly closed nuclei. We see in Table II how everything hangs together. We can explain the reduction in $B(M1)$ in ^{10}Be relative to ^8Be by the fact that the isovector $B(E2)$ in ^{10}Be is much larger than in ^8Be . Note that the isoscalar $B(E2)$'s are almost the same in these two nuclei. The summed $B(M1)$ in ^8Be is $2/\pi$, but in ^{10}Be it is only $3/8$ of that.

In ^6Be and ^{12}Be , the $E2$ transition is from two protons with $L = 0$ $S = 0$ to two protons with $L = 2$ $S = 0$. Note that, surprisingly, the coefficients in front of the effective charge factors is larger for singly-magic ^6Be than it is for the open shell nucleus ^8Be . The

factors are respectively $12.5b^4/\pi$ and $8.75b^4/\pi$. However, the charge factor for ${}^6\text{Be}$ (${}^{12}\text{Be}$) is e_p^2 , whereas for ${}^8\text{Be}$ it is $(e_p + e_n)^2$. The latter gives a factor of four enhancement for the isoscalar $B(E2)$ in ${}^8\text{Be}$.

Again we see from Table II that the isoscalar and isovector $B(E2)$'s are necessarily the same and, when this is fed into the sum rule of Zheng and Zamick [4], one gets the consistent result that $B(M1)_{\text{orbital}}$ is zero.

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TABLES

TABLE I. The Expectation Value of $U/\bar{\chi} = -4(r/b)^4$ for several single-particle states

state	$\langle U/\bar{\chi} \rangle$
$0s$	-15
$0p$	-35
$0d$	-63
$1s$	-75
$0f$	-99
$1p$	-119

TABLE II. The Values of $B(M1)_{orbital}$ and $B(E2)_{isoscalar}$ and $B(E2)_{isovector}$ for Be isotopes.

Nucleus	$B(M1)_{orbital}$	$B(E2)_{isoscalar} (e^2 fm^4)^a$	$B(E2)_{isovector} (e^2 fm^4)^a$
6Be	0	19.92 ^b	19.92 ^b
8Be	$2/\pi=0.637$	82.66 ^c	7.371
${}^{10}Be$	$T = 1 \rightarrow T = 1$ $9/32\pi=0.0895$	70.78 ^d	32.35 ^e
	$T = 1 \rightarrow T = 2$ $15/32\pi=0.149$	0	3.322
${}^{12}Be$	0	19.92	19.92

^aThe value $b = 1.650 fm$ was used for all nuclei above.

^bThe analytic expression in 6Be is $B(E2) = \frac{50}{4\pi}b^4e_p^2$.

^cThe analytic expression in 8Be is $B(E2) = \frac{35}{4\pi}b^4(e_p + e_n)^2$.

^d $B(E2)_{isoscalar} = 0$ to the 2_1^+ state, and is equal to $68.24 e^2 fm^4$ to the 2_2^+ state.

^e $B(E2)_{isovector} = 31.19e^2 fm^4$ to the 2_1^+ state, and is equal to zero for the 2_2^+ state.